

Asymptotic Behaviour of the Conjugacy Probability of the Alternating Group

Misja F.A. Steinmetz

Madeleine L. Whybrow

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Abstract

For G a finite group, $\kappa(G)$ is the probability that $\sigma, \tau \in G$ are conjugate, when σ and τ are chosen independently and uniformly at random. Recently, Blackburn *et al* (2012) gave an elementary proof that $\kappa(S_n) \sim A/n^2$ as $n \rightarrow \infty$ for some constant A - a result which was first proved by Flajolet *et al* (2006). In this paper, we extend the elementary methods of Blackburn *et al* to show that $\kappa(A_n) \sim B/n^2$ as $n \rightarrow \infty$ for some constant B , given explicitly in this paper.

1 Introduction

Let G be a finite group and $\kappa(G)$ be the probability that $\sigma, \tau \in G$ are conjugate, when σ and τ are chosen independently and uniformly at random. In words, $\kappa(G)$ is the probability that two randomly chosen elements from the group are conjugate.

If G is any group and g_1, g_2, \dots, g_k is a complete set of representatives for the conjugacy classes of G , then it is easy to see that

$$\kappa(G) = \frac{1}{|G|^2} \sum_{i=1}^k |g_i^G|^2 = \sum_{i=1}^k \frac{1}{|\text{Cent}_G(g_i)|^2},$$

where $\text{Cent}_G(g)$ denotes the centralizer of an element $g \in G$.

It was first proved by Flajolet *et al* (2006) and later, using more elementary methods, by Blackburn *et al* (2012) that $\kappa(S_n) \sim A/n^2$ as $n \rightarrow \infty$ for some constant A . In this paper, we extend the methods used by Blackburn *et al* (2012, pp. 11-17) to show that $\kappa(A_n) \sim B/n^2$ as $n \rightarrow \infty$ for some constant B .

Although parts of this paper use similar methods to those of Blackburn *et al*, some complications arise when treating the alternating groups instead of the symmetric groups. In particular, a conjugacy class of S_n splits into two classes in A_n if its corresponding cycle type consists of cycles of distinct odd lengths.

In addressing this problem, we introduce the following definitions:

Definition 1.1. Suppose $\sigma, \tau \in S_n$ are chosen independently and uniformly at random. Let

- $\kappa_E(S_n)$ be the probability that σ, τ are conjugate, given that they are even permutations,
- $\kappa_O(S_n)$ be the probability that σ, τ are conjugate, given that they are odd permutations,
- $Q(S_n)$ be the probability that σ, τ have the same cycle type and they are only composed of cycles of distinct odd lengths.

In the remainder of this paper we shall adopt the standard convention that S_0 equals the trivial group. One should note, however, that $\kappa_O(S_n)$ is not well defined for $n = 0, 1$, since with our convention neither S_0 nor S_1 contains any odd permutations. In order for Proposition 2.2 in Section 2 to make sense, we define $\kappa_O(S_0) := 1$ and $\kappa_O(S_1) := 0$. Please note that all of the three statistics given above are now well defined on S_n for all $n \geq 0$.

We find that the asymptotic behavior of $\kappa(A_n)$ depends on the parity of n . We thus split our result into two cases, the first when the limit of $n^2\kappa(A_n)$ is taken over the even integers and the second when it is taken over the odd integers.

We now state our main result:

Theorem 1.2. *Using notation as defined above,*

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa(A_n) = \sum_{\substack{d=0 \\ d \text{ even}}}^{\infty} \kappa_O(S_d) + \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} (\kappa_E(S_d) - 2Q(S_d))$$

and

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 \kappa(A_n) = \sum_{\substack{d=0 \\ d \text{ even}}}^{\infty} (\kappa_E(S_d) - 2Q(S_d)) + \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \kappa_O(S_d).$$

The remainder of this paper concerns the proof of this result. In Section 2 we prove that for $\kappa_E(S_n), \kappa_O(S_n)$ and $Q(S_n)$ we can find inequalities iteratively relating their value to sums over values of these statistics at S_k for smaller k .

We then use these inequalities in Section 3 to find constants C_0, C_1 and C_2 such that for all $n \in \mathbb{N}$

$$\kappa_E(S_n) \leq \frac{C_0}{n^2}, \kappa_O(S_n) \leq \frac{C_1}{n^2} \text{ and } Q(S_n) \leq \frac{C_2}{n^2}.$$

In other words, we will establish uniform bounds on our probabilities in this section.

In Section 4 we find expressions for the limits

$$\lim_{n \rightarrow \infty} n^2 \kappa_E(S_n) \text{ and } \lim_{n \rightarrow \infty} n^2 Q(S_n).$$

Finally in Section 5 we show that we can use our results for $\kappa_E(S_n), \kappa_O(S_n)$ and $Q(S_n)$ to understand the asymptotic behaviour of $\kappa(A_n)$ and prove Theorem 1.2.

2 Inequalities on $\kappa_E(S_n)$ and $Q(S_n)$

In this section we will obtain analogous results to Proposition 7.1 of Blackburn *et al* (2012, p. 13) for $\kappa_E(S_n)$ and $Q(S_n)$. We first state the following well known lemma found in Blackburn *et al* (2012).

Lemma 2.1. *Let $n \in \mathbb{N}$ and let $1 \leq l \leq n$. Let $X \in \Omega_n$ be an l -set. If σ is chosen uniformly and at random from S_n then*

- *the probability that σ acts as an l -cycle on X is $\frac{1}{l} \binom{n}{l}^{-1}$;*
- *the expected number of l -cycles in σ is $1/l$;*
- *the probability that 1 is contained in an l -cycle is $1/n$.*

First we will turn our attention to $\kappa_E(S_n)$. We write $s_k(n)$ for the probability that a permutation of Ω_n , chosen at random, has only cycles of length strictly less than k .

Proposition 2.2. *For all $n \in \mathbb{N}$ we have*

$$\kappa_E(S_n) \leq s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2}.$$

Moreover, if k is such that $n/2 < k \leq n$ then

$$\kappa_E(S_n) \geq \sum_{\substack{l=k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2}.$$

Proof. Let σ and τ be even permutations of Ω_n chosen independently and uniformly at random. Let X and Y be two l -sets and write \overline{X} and \overline{Y} for their respective complements in Ω_n .

Let $E(X, Y)$ be the event that σ acts as an l -cycle on X , τ acts as an l -cycle on Y and that $\overline{\sigma}$ and $\overline{\tau}$, the respective restrictions of σ and τ to \overline{X} and \overline{Y} , have the same cycle structure.

If l is even then $\overline{\sigma}$ and $\overline{\tau}$ must be odd so the probability that they are of the same cycle type is equal to $\kappa_O(S_{n-l})$. If l is odd, the probability is $\kappa_E(S_{n-l})$. So, from Lemma 2.1, we have

$$\mathbf{P}(E(X, Y)) = \begin{cases} \binom{n}{l}^{-2} \frac{\kappa_O(S_n)}{l^2}, & l \text{ even;} \\ \binom{n}{l}^{-2} \frac{\kappa_E(S_n)}{l^2}, & l \text{ odd.} \end{cases}$$

If σ and τ are conjugate in S_n then either σ and τ both contain only cycles of length strictly less than k , or there exists sets X and Y of cardinality $l \geq k$ on which σ and τ act as l -cycles and such that the restrictions $\overline{\sigma}$ and $\overline{\tau}$ have the

same cycle type. Therefore

$$\begin{aligned}
\kappa_E(S_n) &\leq s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ even}}}^n \sum_{|X|=l} \sum_{|Y|=l} \mathbf{P}(E(X, Y)) + \sum_{\substack{l=k \\ l \text{ odd}}}^n \sum_{|X|=l} \sum_{|Y|=l} \mathbf{P}(E(X, Y)) \\
&= s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ even}}}^n \sum_{|X|=l} \sum_{|Y|=l} \binom{n}{l}^{-2} \frac{\kappa_O(S_{n-l})}{l^2} \\
&\quad + \sum_{\substack{l=k \\ l \text{ odd}}}^n \sum_{|X|=l} \sum_{|Y|=l} \binom{n}{l}^{-2} \frac{\kappa_E(S_{n-l})}{l^2} \\
&= s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2}.
\end{aligned}$$

Here the events $E(X, Y)$ are not necessarily disjoint so we have only an upper bound, not an equality. We have established the first inequality of the proposition.

When $k > \frac{n}{2}$ the events $E(X, Y)$ with $|X|, |Y| \geq k$ are disjoint, since a permutation of length n can only contain at most one cycle of length greater than $\frac{n}{2}$. Thus

$$\begin{aligned}
\kappa_E(S_n) &= s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2} \\
&\geq \sum_{\substack{l=k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2},
\end{aligned}$$

as required. \square

We now take a closer look at $Q(S_n)$. Recall that $Q(S_n)$ is the probability that two elements of S_n , chosen independently and uniformly at random, have the same cycle type and consist of cycles of distinct odd lengths.

Proposition 2.3. *For all $n \in \mathbb{N}$ we have*

$$Q(S_n) \leq s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2}.$$

Moreover, if k is such that $\frac{n}{2} < k \leq n$, then

$$Q(S_n) \geq \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2}.$$

Proof. Let l be odd and let $\sigma, \tau \in S_n$. We define $X, Y, \overline{X}, \overline{Y}, \overline{\sigma}$ and $\overline{\tau}$ as in the proof of Proposition 2.2.

Let $F(X, Y)$ be defined as the event that σ acts as an l -cycle on X , τ acts as an l -cycle on Y , $\overline{\sigma}$ and $\overline{\tau}$ have the same cycle type and $\overline{\sigma}$ and $\overline{\tau}$ only have cycles of distinct odd length. Given that σ and τ act as l -cycles on X and Y , $\overline{\sigma}$ and $\overline{\tau}$ are independently and uniformly distributed over the symmetric groups of \overline{X} and \overline{Y} . Hence the probability that $\overline{\sigma}$ and $\overline{\tau}$ have the same cycle type and $\overline{\sigma}$ and $\overline{\tau}$ only have cycles of distinct odd length is exactly $Q(S_{n-l})$. So, from Lemma 2.1,

$$\mathbf{P}(F(X, Y)) = \binom{n}{l}^{-2} \frac{Q(S_{n-l})}{l^2}.$$

If σ and τ have the same cycle type and only parts of distinct odd lengths, then either both have only cycles of length strictly less than k , or there exist l -sets X and Y for some $l \geq k$ such that σ acts on X as an l -cycle and τ acts on Y as an l -cycle and such that the restrictions $\overline{\sigma}$ and $\overline{\tau}$ have the same cycle type and only parts of distinct odd lengths. Therefore

$$\begin{aligned} Q(S_n) &\leq s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ odd}}}^n \sum_{|X|=l} \sum_{|Y|=l} \mathbf{P}(F(X, Y)) \\ &= s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ odd}}}^n \sum_{|X|=l} \sum_{|Y|=l} \binom{n}{l}^{-2} \frac{Q(S_{n-l})}{l^2} \\ &= s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2}. \end{aligned}$$

Here, again, the $F(X, Y)$ are not necessarily independent, meaning we do not have equality. The inequality also allows us to ignore the case where $\overline{\sigma}$ contains a further l -cycle. We have established the first inequality of the proposition.

When $k > \frac{n}{2}$ the events $F(X, Y)$ with $|X|, |Y| \geq k$ are disjoint, since a permutation of length n can only contain at most one cycle of length greater than $\frac{n}{2}$. What's more, $\overline{\sigma}$ may not contain an l -cycle. Thus

$$\begin{aligned} Q(S_n) &= s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} \\ &\geq \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2}. \end{aligned}$$

□

3 Uniform Bounds on $\kappa_E(S_n)$, $\kappa_O(S_n)$ and $Q(S_n)$

In this section we establish an analogous result to Theorem 1.4 in Blackburn *et al* (2012, p. 3). We will prove that there exist C_0 , C_1 and C_2 such that for all $n \in \mathbb{N}$

$$\kappa_E(S_n) \leq \frac{C_0}{n^2}, \kappa_O(S_n) \leq \frac{C_1}{n^2}, Q(S_n) \leq \frac{C_2}{n^2}.$$

We will need the following results, found in Blackburn *et al* (2012, pp. 13–14).

Proposition 3.1. *Let $k \in \mathbb{N}$ be such that $k \geq 2$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$ns_k(n) \geq (n+1)s_k(n)(n+1) \text{ for } n \in \{n_0, n_0+1, \dots, n_0+k-2\}.$$

Then $ns_k(n) \geq (n+1)s_k(n+1)$ for all $n \geq n_0$.

Lemma 3.2. *Let $n \in \mathbb{N}$ and let $0 < n < k/2$. Then*

$$\sum_{l=\lceil n/2 \rceil}^{n-k-1} \frac{1}{l^2(n-l)^2} \leq \frac{1}{n^2k} + \frac{2 \log(n/k)}{n^3}.$$

For the sake of brevity we have decided to omit proofs of these results. They can be found in Blackburn *et al* (2012, pp. 13–15).

We are now ready to start showing that bounds as given above exist. We first consider $\kappa_E(S_n)$ and $\kappa_O(S_n)$. For all $n \geq 2$,

$$\begin{aligned} \kappa_E(S_n) + \kappa_O(S_n) &= 4\kappa(S_n), \\ 0 &\leq \kappa_E(S_n), \kappa_O(S_n) \leq 1 \text{ and} \\ n^2\kappa(S_n) &\leq C_\kappa, \end{aligned}$$

where $C_\kappa := 13^2\kappa(S_{13})$ (Blackburn *et al*, 2012, p. 15). Hence, we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned} n^2\kappa_E(S_n) &\leq 4C_\kappa \text{ and} \\ n^2\kappa_O(S_n) &\leq 4C_\kappa, \end{aligned}$$

which establishes the required upper bounds for $\kappa_E(S_n)$ and $\kappa_O(S_n)$.

We now consider the case of $Q(S_n)$. In this case, we can do better; we may follow the same argument as Blackburn *et al* (2012, pp. 14–16) to find a tight upper bound for $n^2Q(S_n)$. In fact, we will prove that this upper bound is achieved at $n = 4$. We first need a result similar to Lemma 8.2 in Blackburn *et al* (2012, p. 15). We define $C_2 := 4^2Q(S_4)$.

Lemma 3.3. *We have*

(i) $Q(S_n) \leq C_2/n^2$ for all $n \leq 300$;

$$(ii) \sum_{\substack{m=0 \\ m \text{ even}}}^{15} Q(S_m) = \frac{630468719}{521756235} < 1.20836;$$

$$(iii) \sum_{\substack{m=1 \\ m \text{ odd}}}^{15} Q(S_m) = \frac{4429844723}{3652293645} < 1.21290;$$

$$(iv) C_2 = \frac{16}{9} < 1.77778.$$

Proof. Most of these results are straightforward calculations. To prove part (i) we introduce the generating function

$$1 + \sum_{n=1}^{\infty} Q(S_n)x^n = \prod_{d \text{ odd}} \left(1 + \frac{x^d}{d^2}\right).$$

We easily see that this is indeed the required generating function as the coefficient of x^n after multiplying out the brackets will be $\sum_S \left(\prod_{s \in S} \frac{1}{s^2}\right)$ where the sum over all sets S such that its elements are distinct odd integers and $\sum_{s \in S} s = n$. Parts (ii), (iii) and (iv) are straightforward calculations. \square

Two further results can be found in Lemma 8.2 of Blackburn *et al* (2012, p. 15):

$$(i) 60s_{15}(60) = \frac{158929798034197186400893117108816122671}{83317523526667097802976844202788608000} < 0.19076$$

$$(ii) ns_{15}(n) \geq (n+1)s_{15}(n+1) \text{ for } 14 \leq n \leq 60.$$

We can now prove our main proposition in a similar way to the proof of Theorem 1.4 in Blackburn *et al* (2012, pp. 15-16).

Proposition 3.4. $Q(S_n) \leq C_2/n^2$ for all $n \in \mathbb{N}$. Moreover, since we defined $C_2 := 4^2Q(S_4)$, this bound is tight.

Proof. We proceed by induction on n . By Lemma 3.3(i) the proposition holds if $n \leq 300$, and so we may conclude that $n > 300$. By Proposition 2.2 in the case $k = 15$ we have

$$Q(S_n) \leq s_{15}(n)^2 + \sum_{\substack{l=15 \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2}$$

and hence

$$n^2Q(S_n) \leq n^2s_{15}(n)^2 + n^2 \sum_{\substack{l=n-15 \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} + n^2 \sum_{\substack{l=15 \\ l \text{ odd}}}^{n-16} \frac{Q(S_{n-l})}{l^2}. \quad (1)$$

It follows from Proposition 3.1 and the comments preceding this proposition that $ns_{15}(n) \leq 60s_{15}(60) < 0.19076$. Hence $n^2s_{15}(n)^2 \leq 0.03639$.

Using Lemma 3.3(ii) to bound the second term in (1), we get

$$\begin{aligned}
n^2 \sum_{\substack{l=n-15 \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} &\leq \left(\frac{n}{n-15}\right)^2 \sum_{\substack{m=1 \\ m \text{ odd}}}^{15} Q(S_m) \\
&\leq \left(\frac{300}{285}\right)^2 \sum_{\substack{m=1 \\ m \text{ odd}}}^{15} Q(S_m) \\
&\leq 1.34393.
\end{aligned}$$

It is clear from Lemma 3.3(ii) and (iii) that the sum over odd values of $Q(S_n)$ is the larger of the two values, and thus the one we take for our bound.

For the third term in (1) we use the inductive hypothesis to get

$$n^2 \sum_{\substack{l=15 \\ l \text{ odd}}}^{n-16} \frac{Q(S_{n-l})}{l^2} \leq n^2 \sum_{\substack{l=15 \\ l \text{ odd}}}^{n-16} \frac{C_2}{l^2(n-l)^2}.$$

Using the symmetry of $l^2(n-l)^2$ in this sum and then applying Lemma 3.2 in the case $n = 15$, we get

$$\begin{aligned}
n^2 \sum_{\substack{l=15 \\ l \text{ odd}}}^{n-16} \frac{1}{l^2(n-l)^2} &\leq n^2 \sum_{l=15}^{n-16} \frac{1}{l^2(n-l)^2} \\
&\leq 2n^2 \sum_{\lceil n/2 \rceil}^{n-16} \frac{1}{l^2(n-l)^2} + \frac{n^2}{15^2(n-15)^2} \\
&\leq 2 \left(\frac{1}{15} + \frac{2 \log(n/15)}{n} \right) + \frac{1}{15^2} \left(\frac{300}{285} \right)^2.
\end{aligned}$$

Since $\log(n/15)/n$ is decreasing for $n > 40$, it follows from the upper bound for C_2 in Lemma 3.3(iv) that

$$n^2 \sum_{\substack{l=15 \\ l \text{ odd}}}^{n-16} \frac{Q(S_{n-l})}{l^2} \leq C_2 \left(\frac{2}{15} + \frac{4 \log(300/15)}{300} + \frac{300^2}{15^2 \cdot 285^2} \right) \leq 0.31681.$$

Hence

$$n^2 Q(S_n) \leq 0.03639 + 1.34393 + 0.31681 = 1.69713 < C_2$$

and the proposition follows. \square

4 Limits

In this section we will prove asymptotic results on $\kappa_E(S_n)$, $Q(S_n)$ and $\kappa(A_n)$ analogous to Theorem 1.5 in Blackburn *et al* (2012, p. 3).

We will need the following lemma, found in Blackburn *et al* (2012, p. 12). Recall that we write $s_k(n)$ for the probability that a permutation of Ω_n , chosen at random, has only cycles of length strictly less than k .

Lemma 4.1. *For all $n, k \in \mathbb{N}$ with $k \geq 2$ we have $s_k(n) \leq 1/t! < \left(\frac{e}{t}\right)^t$ when $t = \lfloor n/(k-1) \rfloor$.*

For brevity, we introduce the following notation:

$$\begin{aligned} A_1 &= \sum_{\substack{d=0 \\ d \text{ even}}}^{\infty} \kappa_O(S_d) + \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \kappa_E(S_d); \\ A_2 &= \sum_{\substack{d=0 \\ d \text{ even}}}^{\infty} \kappa_E(S_d) + \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \kappa_O(S_d); \\ B_1 &= \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} Q(S_d); \\ B_2 &= \sum_{\substack{d=0 \\ d \text{ even}}}^{\infty} Q(S_d). \end{aligned}$$

Proposition 4.2.

$$\begin{aligned} \liminf_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa_E(S_n) &\geq A_1, \\ \liminf_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 \kappa_E(S_n) &\geq A_2. \end{aligned}$$

Proof. The second part of Proposition 2.2 says that, if $k > n/2$ then

$$\kappa_E(S_n) \geq \sum_{\substack{l=k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2}.$$

Hence

$$n^2 \kappa_E(S_n) \geq \sum_{\substack{l=k \\ l \text{ even}}}^n \kappa_O(S_{n-l}) + \sum_{\substack{l=k \\ l \text{ odd}}}^n \kappa_E(S_{n-l}).$$

Taking $k = \lfloor 3n/4 \rfloor$ and letting $n \rightarrow \infty$ we see that

$$\begin{aligned} \liminf_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa_E(S_n) &\geq \sum_{\substack{d=0 \\ d \text{ even}}}^{\infty} \kappa_O(S_d) + \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \kappa_E(S_d) = A_1, \\ \liminf_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 \kappa_E(S_n) &\geq \sum_{\substack{d=0 \\ d \text{ even}}}^{\infty} \kappa_E(S_d) + \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \kappa_O(S_d) = A_2. \end{aligned}$$

□

Proposition 4.3.

$$\limsup_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa_E(S_n) \leq A_1,$$

$$\limsup_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 \kappa_E(S_n) \leq A_2.$$

Proof. Let $k = \lfloor \frac{n}{\log(n)} \rfloor$. By Proposition 2.2 we have

$$\kappa_E(S_n) \leq s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2}.$$

By Lemma 4.1 we have

$$s_k(n) < \left(\frac{e}{t}\right)^t$$

where $t = \lfloor \frac{n}{k-1} \rfloor$. Writing $k = n/\log n + O(1)$ we have

$$\lfloor \frac{n}{k-1} \rfloor = (\log n) \left(1 + O\left(\frac{\log n}{n}\right)\right),$$

and so

$$\begin{aligned} \log(ns_k(n)) &< \log(n)t(1 - \log t) \\ &= \log n - \log n \log \log n + \log \left(1 + O\left(\frac{\log n}{n}\right)\right) \\ &\rightarrow -\infty \end{aligned}$$

as $n \rightarrow \infty$. Hence $ns_k(n) \rightarrow 0$ as $n \rightarrow \infty$.

We estimate the main sum in the same way as in Blackburn *et al* (2012, p. 17). This gives

$$\begin{aligned} &\sum_{\substack{l=k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2} \\ &\leq \sum_{\substack{l=n-k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ even}}}^{n-k-1} \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^{n-k-1} \frac{\kappa_E(S_{n-l})}{l^2} \\ &\leq \sum_{\substack{l=n-k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2} + \sum_{l=k}^{n-k-1} \frac{C'}{l^2(n-l)^2} \end{aligned} \quad (2)$$

$$\leq \sum_{\substack{l=n-k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2} + \sum_{l=\lceil n/2 \rceil}^{n-k-1} \frac{2C'}{l^2(n-l)^2} + \frac{C'}{k^2(n-k)^2}, \quad (3)$$

where equation (2) is justified by the bounds in Section 3, C' is defined as $\max\{C_0, C_1\}$ and equation (3) is as a result of the symmetry in $l^2(n-l)^2$.

By Lemma 3.2 the third term in equation (3) is at most $(2C' \log n)/n^3$. Moreover, from the identity $\frac{n}{k(n-k)} = \frac{1}{k} + \frac{1}{n-k}$ it is clear that $\frac{n^2}{k^2(n-k)^2} \rightarrow 0$ as $n \rightarrow \infty$. So the last two terms in equation (3) are $o(n^{-2})$ and may safely be ignored.

We now consider the case when n is even. Let $\epsilon \in \mathbb{R}$ be such that $0 < \epsilon < 1$. For all n such that $1/\log(n) < \epsilon$ we have

$$m/n \leq (n/\log(n))/n < \epsilon$$

and thus

$$\begin{aligned} n^2 \left(\sum_{\substack{l=n-k \\ l \text{ even}}}^n \frac{\kappa_O(S_{n-l})}{l^2} + \sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{\kappa_E(S_{n-l})}{l^2} \right) \\ = n^2 \left(\sum_{\substack{m=0 \\ m \text{ even}}}^k \frac{\kappa_O(S_m)}{(n-m)^2} + \sum_{\substack{m=1 \\ m \text{ odd}}}^k \frac{\kappa_E(S_m)}{(n-m)^2} \right) \\ = \sum_{\substack{m=0 \\ m \text{ even}}}^k \frac{\kappa_O(S_m)}{(1-m/n)^2} + \sum_{\substack{m=1 \\ m \text{ odd}}}^k \frac{\kappa_E(S_m)}{(1-m/n)^2} \\ \leq \frac{1}{(1-\epsilon)^2} \left(\sum_{\substack{m=0 \\ m \text{ even}}}^k \kappa_O(S_m) + \sum_{\substack{m=1 \\ m \text{ odd}}}^k \kappa_E(S_m) \right). \end{aligned}$$

These remarks give us

$$\begin{aligned} \limsup_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa_E(S_n) &\leq \limsup_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa_E(S_n) \\ &\leq \frac{1}{(1-\epsilon)^2} \left(\sum_{\substack{m=0 \\ l \text{ even}}}^{\lfloor n/\log(n) \rfloor} \kappa_O(S_m) + \sum_{\substack{m=1 \\ l \text{ odd}}}^{\lfloor n/\log(n) \rfloor} \kappa_E(S_m) \right) \\ &\leq \frac{1}{(1-\epsilon)^2} \left(\sum_{\substack{m=0 \\ l \text{ even}}}^{\infty} \kappa_O(S_m) + \sum_{\substack{m=1 \\ l \text{ odd}}}^{\infty} \kappa_E(S_m) \right) \\ &= \frac{A_1}{(1-\epsilon)^2}. \end{aligned}$$

But as ϵ was arbitrary, we conclude that

$$\limsup_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa_E(S_n) \leq A_1.$$

Similarly, we find that

$$\limsup_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 \kappa_E(S_n) \leq A_2,$$

which completes the proof. \square

From Propositions 4.2 and 4.3 we can now conclude that

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa_E(S_n) = A_1$$

and

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 \kappa_E(S_n) = A_2.$$

This concludes the first half of this section. The second half of this section will focus on $Q(S_n)$ using a similar approach to that of the first half.

Proposition 4.4.

$$\liminf_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 Q(S_n) \geq B_1,$$

$$\liminf_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 Q(S_n) \geq B_2.$$

Proof. The second half of Proposition 2.3 tells us that if k is such that $\frac{n}{2} < k \leq n$, then

$$Q(S_n) \geq \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} = \begin{cases} \sum_{\substack{m=1 \\ m \text{ odd}}}^{n-k} \frac{Q(S_m)}{(n-m)^2}, & \text{for } n \text{ even,} \\ \sum_{\substack{m=0 \\ m \text{ even}}}^{n-k} \frac{Q(S_m)}{(n-m)^2}, & \text{for } n \text{ odd.} \end{cases}$$

Hence the result follows in the same way as in Proposition 4.2. \square

Proposition 4.5.

$$\limsup_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 Q(S_n) \leq B_1,$$

$$\limsup_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 Q(S_n) \leq B_2.$$

Proof. As before, we let $k = \left\lfloor \frac{n}{\log(n)} \right\rfloor$. By Proposition 2.3

$$Q(S_n) \leq s_k(n)^2 + \sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2}.$$

We already know that $ns_k \rightarrow 0$ as $n \rightarrow \infty$ and we estimate the remaining sum as before:

$$\begin{aligned}
\sum_{\substack{l=k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} &= \sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} + \sum_{\substack{l=k \\ l \text{ odd}}}^{n-k-1} \frac{Q(S_{n-l})}{l^2} \\
&\leq \sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} + \sum_{l=k}^{n-k-1} \frac{Q(S_{n-l})}{l^2} \\
&\leq \sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} + \sum_{l=k}^{n-k-1} \frac{C_2}{l^2(n-l)^2} \\
&\leq \sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} + \sum_{l=\lceil n/2 \rceil}^{n-k-1} \frac{2C_2}{l^2(n-l)^2} + \frac{C_2}{k^2(n-k)^2}.
\end{aligned}$$

As in Proposition 4.3, we find that everything in this sum is $o(n^{-2})$ except the term

$$\sum_{\substack{l=n-k \\ l \text{ odd}}}^n \frac{Q(S_{n-l})}{l^2} = \begin{cases} \sum_{\substack{m=1 \\ m \text{ odd}}}^k \frac{Q(S_m)}{(n-m)^2}, & \text{for } n \text{ is even,} \\ \sum_{\substack{m=0 \\ m \text{ even}}}^k \frac{Q(S_m)}{(n-m)^2}, & \text{for } n \text{ is odd.} \end{cases}$$

Hence, similarly to before, we find that for an arbitrary $0 < \epsilon < 1$

$$\limsup_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 Q(S_n) \leq \frac{B_1}{(1-\epsilon)^2},$$

$$\limsup_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 Q(S_n) \leq \frac{B_2}{(1-\epsilon)^2}$$

and the proposition follows since ϵ is arbitrary. \square

From Proposition 4.4 and 4.5 we can now conclude that

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 Q(S_n) = B_1,$$

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 Q(S_n) = B_2.$$

5 Asymptotics of $\kappa(A_n)$

Lemma 5.1. *We have*

$$\kappa(A_n) = \kappa_E(S_n) - 2Q(S_n).$$

Proof. Let $\sigma, \tau \in S_n$ be chosen uniformly and independently at random. Let

- AN be the event that both σ and τ lie in A_n ,
- CT be the event that they have the same cycle type,
- SP be the event that they lie in a conjugacy class in A_n which is split, i.e. that they are formed only of parts of odd, distinct length.

We note that $\kappa_E(S_n) = \mathbf{P}(CT|AN)$ and that $Q(S_n) = \mathbf{P}(CT \cap SP)$. Moreover,

$$\kappa(A_n) = \mathbf{P}(CT|AN) \times \left(1 - \frac{1}{2} \mathbf{P}(SP|AN \cap CT)\right)$$

for if the conjugacy class is not split, then they are automatically conjugate in A_n when they have the same cycle type, and if the conjugacy class is split, then the chance they are conjugate is $\frac{1}{2}$ when they have the same cycle type. We have

$$\mathbf{P}(SP|AN \cap CT) = \frac{\mathbf{P}(SP \cap CT|AN)}{\mathbf{P}(CT|AN)} = \frac{\mathbf{P}(SP \cap CT|AN)}{\kappa_E(S_n)}.$$

But SP implies AN, hence

$$\mathbf{P}(SP \cap CT|AN) = \frac{\mathbf{P}(SP \cap CT \cap AN)}{\mathbf{P}(AN)} = \frac{\mathbf{P}(SP \cap CT)}{\frac{1}{4}} = 4Q(S_n).$$

Thus

$$\mathbf{P}(SP|AN \cap CT) = \frac{4Q(S_n)}{\kappa_E(S_n)}$$

and

$$\kappa(A_n) = \kappa_E(S_n) - 2Q(S_n).$$

□

Proof of Theorem 1.2. From the above lemma, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} n^2 \kappa(A_n) &= \lim_{n \rightarrow \infty} \{n^2 \kappa_E(S_n) - 2n^2 Q(S_n)\} \\ &= \lim_{n \rightarrow \infty} \{n^2 \kappa_E(S_n)\} - 2 \lim_{n \rightarrow \infty} \{n^2 Q(S_n)\} \end{aligned}$$

and we can finally conclude that

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ n \text{ even}}} n^2 \kappa(A_n) &= A_1 - 2B_1, \\ \lim_{\substack{n \rightarrow \infty \\ n \text{ odd}}} n^2 \kappa(A_n) &= A_2 - 2B_2. \end{aligned}$$

□

6 Further Remarks

It is clear that there is still more which can be said about the conjugacy probability of A_n , and of other groups. Firstly, it is possible that the methods used by Flajolet *et al* (2006) would allow the explicit calculation the asymptotic values of $\kappa(A_n)$ to greater precision.

It may also be interesting to consider the conjugacy probability to other families of groups. In particular, the family $GL(n, q)$, with q fixed and n tending to infinity may be an interesting case to consider.

Lastly, a different but related probability that may be of interest is the probability that one element chosen uniformly at random from S_n belongs to a split conjugacy class in A_n - we called this probability $q(S_n)$. Numerical evidence suggests $n^{-\frac{1}{2}}q(S_n)$ has a limit as n tends to infinity but our brief investigations suggest that this would be harder to prove than the limits treated in this paper.

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